Today

1. Differential equations
2. Simulation diagrams
3. Systems described by differential equations
4. The impulse response
Definition

A linear differential equation of order $n$ is a differential equation of the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t),$$

where $a_k(t)$ and $g(t)$ are functions of $t$, and $a_n(t) \neq 0$.

- The functions $a_k(t)$ are called coefficients.
- A linear differential equation with constant coefficients is of the form:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t),$$

where $a_k$ is constant and $a_n \neq 0$.
- We often assume the equation is monic, that is: $a_n = 1$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t).$$

The complementary equation

Definition

- A linear differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t) \quad (*)$$

is called homogeneous if $g(t) = 0$.

- If $g(t) \neq 0$, then the equation is called inhomogeneous.

- If $(*)$ is inhomogeneous, then the equation

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$$

is called the complementary equation of $(*)$.

- The solution of the complementary equation is called the complementary solution.
### Recap

#### Theorem

*For every $n$-th order homogeneous linear differential equation there exists solutions $y_1(t), y_2(t), \ldots, y_n(t)$ such that every solution $y(t)$ can be written as*

$$y(t) = C_1y_1(t) + C_2y_2(t) + \cdots + C_ny_n(t)$$

*for some constants $C_1, \ldots, C_n$.*

- The constants $C_k$ are called **integration constants**.
- The solutions $y_k(t)$ are called **fundamental solutions**.
- The set of fundamental solutions $y_k(t)$ are not unique.
- The set of fundamental solutions $\{y_1(t), \ldots, y_n(t)\}$ are a basis for the solution space.

#### Inhomogeneous linear differential equations

*Let $y_p(t)$ be a solution of an $n$-th order inhomogeneous linear differential equation*

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = g(t). \quad (\ast)$$

*Then every solution $y(t)$ of (\ast) can be written as*

$$y(t) = y_h(t) + y_p(t),$$

*where $y_h(t)$ is a complementary solution.*

- The solution $y_p(t)$ is called a **particular solution**.
- Every solution $y(t)$ of (\ast) can be written as

$$y(t) = C_1y_1(t) + C_2y_2(t) + \cdots + C_ny_n(t) + y_p(t),$$

*where $y_1(t), \ldots, y_n(t)$ are fundamental solutions of (\ast).*
First-order linear differential equations

\textbf{Theorem}

Let \( y' + a_0 y = g(t) \) be a monic, linear first-order differential equation.

- The complementary solution is
  \[ y_h(t) = Ce^{-a_0 t} \]

- A particular solution is
  \[ y_p(t) = e^{-a_0 t} \int g(t) e^{a_0 \tau} \, d\tau \]

- If the initial condition is \( y(t_0) = y_0 \), then
  \[ y(t) = y_0 e^{a_0 (t_0 - t)} + e^{-a_0 t} \int_{t_0}^{t} g(\tau) e^{a_0 \tau} \, d\tau \]

\textbf{The delta function}

\textbf{Example}

Solve the differential equation \( y' - 2y = \delta(t) \).

- The complementary solution is \( y_h(t) = Ce^{2t} \).
- A particular solution is
  \[ y_p(t) = e^{2t} \int_{-\infty}^{t} \delta(\tau) e^{-2\tau} \, d\tau \]
  sifting property
  \[ = \begin{cases} e^{2t} e^{-2\cdot0} & \text{if } t > 0, \\ 0 & \text{if } t < 0, \end{cases} \]
  \[ = e^{2t} u(t). \]
- The general solution is
  \[ y(t) = e^{2t} (u(t) + C). \]

- Check:
  \[ y'(t) - 2y(t) = e^{2t} \delta(t) \]
  sampling property
  \[ = e^{2t} \delta(t) = \delta(t). \]
Initial conditions

Example

Solve the initial value problem

\[
\begin{align*}
y' - 2y &= \delta(t), \\
y(0^-) &= 0.
\end{align*}
\]

- The general solution is \( y(t) = e^{2t}(u(t) + C) \).
- Because of the jump at 0, initial conditions of the form \( y(0) = y_0 \) give ambiguous results. Hence the use of the one-sided limit in the initial condition.
- Use the initial condition to evaluate \( C \):
  \[
  0 = y(0^-) = \lim_{t \to 0^-} e^{2t}(u(t) + C) = e^{2\cdot0}(0 + C),
  \]
  hence \( C = 0 \).
- Solution: \( y(t) = e^{2t}u(t) \).

Solution:

\[
y(t) = e^{2t}u(t).
\]

Initial conditions

Example

Solve the initial value problem

\[
\begin{align*}
y' - 2y &= \delta(t), \\
y(0^+) &= 0.
\end{align*}
\]

- The general solution is \( y(t) = e^{2t}(u(t) + C) \).
- Use the initial condition to evaluate \( C \):
  \[
  0 = y(0^+) = \lim_{t \to 0^+} e^{2t}(u(t) + C) = e^{2\cdot0}(1 + C),
  \]
  hence \( C = -1 \).
- Solution: \( y(t) = -e^{2t}(1 - u(t)) = -e^{2t}u(-t) \).
Differential equations as linear systems

- A system described by a differential equation is initial value problem of the form

\[
\begin{align*}
F(t, y, y', y'', \ldots, y^{(n)}) &= G(x(t)), \\
y(t_0) &= y_0, \\
y'(t_0) &= y_1, \\
&\vdots \\
y^{(n-1)}(t_0) &= y_{n-1},
\end{align*}
\]

where \(x(t)\) and \(t_0\) are the input of the system.

**Theorem**

A system described by a differential equation is linear if it is of the form

\[
\begin{align*}
& a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y \\
= & b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_0 x(t), \\
& y(t_0) = y'(t_0) = y''(t_0) = \ldots = y^{(n-1)}(t_0) = 0.
\end{align*}
\]

Differential equations and time invariance

- The system described by

\[
\begin{align*}
& a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y \\
= & b_m x^{(m)}(t) + b_{m-1} x^{(m-1)}(t) + \cdots + b_0 x(t), \\
& y(t_0) = y'(t_0) = y''(t_0) = \ldots = y^{(n-1)}(t_0) = 0
\end{align*}
\]

is time-invariant if \(t_0\) shifts accordingly whenever \(x(t)\) shifts: if the input is \(x(t - \alpha)\) then \(y(t - \alpha)\) is a solution whenever the initial condition is replaced by

\[
y(t_0 - \alpha) = y'(t_0 - \alpha) = \ldots = y^{(n-1)}(t_0 - \alpha) = 0.
\]
The impulse response

**Theorem**

Suppose that a particular solution \( y_p(t) \) satisfies the differential equation

\[
  a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = b_m \delta^{(m)}(t) + b_{m-1} \delta^{(m-1)}(t) + \cdots + b_0 \delta(t),
\]

and suppose moreover that \( y_p(t) = 0 \) for all \( t < 0 \), then \( y_p \) is the impulse response.

- Note that \( y_p^{(j)}(t) = 0 \) for all \( t < 0 \).
- From this follows that
  \[
  y_p(0^-) = y'(0^-) = y''(0^-) = \ldots = y^{(n-1)}(0^-) = 0.
  \]

Second-order linear differential equations

**Example**

Find the impulse response of the differential equation

\[
  y'' + 2y' + 2y = x'' + 3x' + 3x.
\]

- The characteristic equation is \( s^2 + 2s + 2 = 0 \), the roots are \(-1 + i\) and \(-1 - i\) respectively.
- The complementary solution is
  \[
  y_h(t) = e^{-t}(C_1 \cos t + C_2 \sin t).
  \]
- Find a particular solution of
  \[
  y'' + 2y' + 2y = \delta''(t) + 3\delta'(t) + 3\delta(t),
  \]
  (1)
  so we first try:
  \[
  y_p(t) = A \delta(t).
  \]
- Note that \( y_p'(t) = A \delta'(t) \) and \( y_p''(t) = A \delta''(t) \), hence equation (1) becomes
  \[
  A \delta''(t) + 2A \delta'(t) + 2A \delta(t) = \delta''(t) + 3\delta'(t) + 3\delta(t).
  \]
- There is no \( A \) for which this equation holds.
Example 2.5.6

The attempt \(y_p(t) = A\delta(t)\) fails, so we try:

\[y_p(t) = A\delta(t) + y_h(t)u(t)\]

Note that there are three (yet to determine) constants.

\[y_p'(t) = A\delta' + y_p' + y_h'u + y_h\delta,\]

\[y_p''(t) = A\delta'' + y_p'' + 2y_p'\delta + y_h\delta'.\]

Substitute \(y_p, y'_p\) and \(y''_p\) in the left-hand side of (1):

\[y''_p + 2y'_p + 2y_p = A(\delta'' + 2\delta' + 2\delta) + (y''_h + 2y'_h + 2y_h)u + (2y'_h + 2y_h)\delta + y_h\delta'.\]

Equation (1) boils down to

\[A\delta'' + (2A + y_h)\delta' + (2A + 2y_h + 2y'_h)\delta = \delta''(t) + 3\delta'(t) + 3\delta(t).\]

From this follows:

\[A = 1, \quad C_1 = 1 \quad \text{and} \quad C_2 = 0.\]

The sought-after particular solution is

\[y_p(t) = \delta(t) + e^{-t} \cos(t) u(t).\]

Since \(y_p(t) = 0\) for \(t < 0\), this is also the impulse response:

\[h(t) = \delta(t) + e^{-t} \cos(t) u(t).\]
System diagrams

- Basic components:

  ![System Diagrams]

  - Adder
  - Subtractor

  The series arrangement of two systems is equivalent to taking the convolution of the impulse responses:

  ![Convolution Diagram]

The first canonical form, first order

- 

  \[
  \begin{align*}
  y &= y_1 + b_1 x(t), \\
  y_1' &= -a_0 y + b_0 x(t), \\
  y' - b_1 x'(t) &= -a_0 y + b_0 x(t), \\
  y' + a_0 y &= b_1 x'(t) + b_0 x(t)
  \end{align*}
  \]

  - Eliminate \( y_1' \)
Example See lecture 3

- The differential equation $y' + y = x(t)$ can be represented by following diagram:

  ![Diagram 1]

- This diagram is equivalent to

  ![Diagram 2]

The first canonical form, second order

- $y = y_1 + b_2 x$,  \[ y'' = y_1'' + b_2 x'' \]
- $y'_1 = y_2 - a_1 y + b_1 x$,  \[ y''_1 = y_2' - a_1 y'_1 + b_1 x' \]
- $y'_2 = -a_0 y + b_0 x$.

- $y'' + a_1 y' + a_0 y = b_2 x'' + b_1 x' + b_0 x$
Example 2.5.2

The differential equation \( y'' - 4y' + y = 4x'(t) + 2x(t) \) is represented by the diagram

- The equation is obtained by eliminating \( y_1 \) from
  \[
  \begin{align*}
  y' &= y_1 + 4y + 4x(t), \\
  y_1' &= -y + 2x.
  \end{align*}
  \]

State equations

- The output signals of the integrators constitute the \textbf{state} of the system, and are called \textbf{state variables}.
- The system is described by the differential equations involving the state variables plus the equation that relates \( y \) to the state variables:
  \[
  \begin{align*}
  v_1' &= -a_1 y + v_2 + b_1 x(t) \\
  v_2' &= -a_0 y + b_0 x(t) \\
  y &= v_1 + b_2 x(t)
  \end{align*}
  \]
The state equations and matrix form

- Eliminating $y$ yield the state equations:
  
  \[
  \begin{align*}
  v'_1 &= -a_1 v_1 + v_2 + (b_1 - a_1 b_2)x(t) \\
  v'_2 &= -a_0 v_1 + (b_0 - a_0 b_2)x(t) \\
  y &= v_1 + b_2 x(t)
  \end{align*}
  \]

- Define vector $\mathbf{v} = (v_1, v_2)$, then
  \[
  \mathbf{v}' = \begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + x(t) \begin{bmatrix} b_1 - a_1 b_2 \\ b_0 - a_0 b_2 \end{bmatrix}.
  \]

- Define $A = \begin{bmatrix} -a_1 & 1 \\ -a_0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 - a_1 b_2 \\ b_0 - a_0 b_2 \end{bmatrix}$, then
  \[
  \mathbf{v}' = A\mathbf{v} + x(t)\mathbf{b}
  \]
  (1)

- Rewrite $y$ in vector form:
  \[
  y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{v} + b_2 x(t)
  \]
  (2)

- (1) and (2) form the state equations in vector form.

Example

Find the state equations for the LTI system described by
\[
2y'' + 4y' + 3y = 4x'(t) + 2x(t).
\]

- Make the equation monic:
  \[
  y'' + 2y' + \frac{3}{2}y = 2x'(t) + x(t).
  \]

- This gives $a_0 = \frac{3}{2}$, $a_1 = 2$, $b_0 = 1$, $b_1 = 2$ and $b_2 = 0$.

- The state equations are
  \[
  \begin{align*}
  v'_1 &= -2v_1 + v_2 + 2x \\
  v'_2 &= -\frac{3}{2}v_1 + x \\
  y &= v_1
  \end{align*}
  \]

- The matrix form is
  \[
  \begin{bmatrix} v' \\ y \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -\frac{3}{2} & 0 \end{bmatrix} \begin{bmatrix} v \\ x(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}
  \]
  \[
  y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{v}
  \]
Notes

- Section 2.6.1 presents an ad-hoc approach where the variables \( v_1(t) = y(t) \) and \( v_2(t) = y'(t) \) are used as state variables. Study example 2.6.1.

- The state equations can be expanded for linear differential equations of arbitrary order.

  You are not required to know this for the test.

- An alternative set of equations can be derived using the second canonical form (section 2.6.4).

  You are not required to know this for the test.

- Solving the state equations (section 2.6.2) is postponed until the end of this course.